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Theorem 1. *Let* $f : [a, b] \to \mathbb{R}$ *be continuous and let* $p^* = \{a = x_0 < \cdots < x_n = b\}$ *be a partition of* $[a, b]$ *where each* x_k *is a nontrivial local extremum of* f*. (We suppose that* f *has only a finite number of local extrema.) Then*

$$\sum_{p^*} |\Delta f_k| = \sup_{p \in \mathcal{P}[a,b]} \sum_{p} |\Delta f_k|$$

Proof. It suffices to prove the result for $p = p^* \cup \{c\}$, where $c \in (x_{k-1}, x_k)$ for some k. Define $p_1 = \{a = x_0 < \cdots < x_k\}$ and $p_2 = \{x_{k+1} < \cdots < x_n = b\}$. Then we have

$$\sum_{p} |\Delta f_k| = \sum_{p_1} |\Delta f_k| + |f(c) - f(x_{k-1})| + |f(x_k) - f(c)| + \sum_{p_2} |\Delta f_k|$$

Since each restriction of f to $[x_{j-1}, x_j]$ is monotonic (because $x_{j-1},\ x_j$ are local extrema) we have that

$$|f(c) - f(x_{k-1})| + |f(x_k) - f(c)| = |f(x_k) - f(x_{k-1})|$$

so that

$$\begin{aligned} \sum_{p} |\Delta f_k| &= \sum_{p_1} |\Delta f_k| + |f(x_k) - f(x_{k-1})| + \sum_{p_2} |\Delta f_k| \\ &= \sum_{p^*} |\Delta f_k| \end{aligned}$$

□

This theorem says, roughly, that there is some "optimal" partition p^* (defined in a precise manner) such that we need never seek a partition finer than p^* when we're finding the total variation of a continuous function. If the function is once differentiable, this theorem becomes particularly useful: differentiate to find the critical points, and then set the partition $p^* = \{x_k : x_k \text{ is a critical point of f}\}$, which will contain all of f's local extrema (and maybe more points).