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Let $u \in \mathbb{R}^d$ be a unit vector. For $x \in \mathbb{R}^d$, define $f(x) = \inf_{t\in\mathbb{R}} ||x - tu||^2$. Show that $f$ is differentiable on all of $\mathbb{R}^d$, and find an expression for $f'(p)$ in terms of $p$ and $u$.

*Proof.* We first find the value of $t$ that satisfies the infimum. Expanding $||x - tu||^2$ gives

$$\begin{aligned}
||x - tu||^2 &= (x - tu) \cdot (x - tu) \\
&= x \cdot x - 2x \cdot (tu) + (tu) \cdot (tu) \\
&= ||x||^2 - 2t(x \cdot u) + t^2||u||^2 \\
&= ||x||^2 - 2t(x \cdot u) + t^2 \qquad \text{because } u \text{ is a unit vector}
\end{aligned}$$

Differentiating, we have

$$\begin{aligned}
\frac{d}{dt}||x - tu||^2 &= -2(x \cdot u) + 2t = 0 \\
2t &= 2(x \cdot u) \\
t &= x \cdot u
\end{aligned}$$

We can thus rewrite $f(x)$ as $f(x) = ||x||^2 - (x \cdot u)^2$. Our task is now to find an expression for the derivative of $f$. We recall that, if $f$ is differentiable at a point $c \in \mathbb{R}^d$, we must have

$$f(c + v) = f(c) = T_c(v) + ||v||E_c(v)$$

where $T_c(v)$ is a linear operator and $E_c(v)$ is some error function such that $E_c(v) \to 0$ as $v \to 0$. Now, since $f : \mathbb{R}^d \to \mathbb{R}$, we know that the only possible candidate for $T_c(v)$ is $\nabla f(c) \cdot v$. Thus, we first compute $f$'s gradient component-wise. Observe that since $f(x) = (x \cdot x) - (x \cdot u)^2 = \sum x_i^2 - (\sum x_i u_i)^2$, we have

$$\frac{\partial f}{\partial x_i} = 2x_i - 2\Big(\sum x_i u_i\Big)u_i \tag{1}$$

so that applying Eq. 1 component-wise and dotting the resulting vector with $v$ gives

$$\nabla f(x) \cdot v = 2(x \cdot v) - 2(x \cdot u)(u \cdot v).$$

Now, from the definition of $f$ we have that, for any $c \in \mathbb{R}^d$,

$$\begin{aligned}
f(c + v) - f(c) &= ||c + v||^2 - ||c||^2 - 2(c \cdot u)(v \cdot u) - (v \cdot u)^2 \\
&= ||c||^2 + 2(c \cdot v) + ||v||^2 - ||c||^2 - 2(c \cdot u)(v \cdot u) - (v \cdot u)^2 \\
&= \underbrace{2(c \cdot v) - 2(c \cdot u)(v \cdot u)}_{T_c(v) = \nabla f(c) \cdot v} + \underbrace{||v||^2 - (v \cdot u)^2}_{R(v)}
\end{aligned}$$

We will now show that the function $R(v)$ has the proper form. Let us rewrite $R(v)$ as

$$R(v) = ||v||^2 - (v \cdot u)^2 = ||v||\Big(||v|| - \frac{(v \cdot u)^2}{||v||}\Big)$$

Obviously $||v|| \to 0$ as $v \to 0$, so all we have to show is that $\frac{(v \cdot u)^2}{||v||} \to 0$ as $||v|| \to 0$. Since $||u|| = 1$, for $||v|| > 0$, letting $\theta$ be the angle between $v$ and $u$ we have that

$$\begin{aligned}
0 \le \frac{(v \cdot u)^2}{||v||} &= \frac{(||v||||u|| \cos\theta)^2}{||v||} \\
&= \frac{||v||^2||u||^2 \cos^2\theta}{||v||} \\
&= ||v|| \cos^2\theta \le ||v||
\end{aligned}$$

So the function $E_c(v)$ has the proper form, and we have that

$$f'(p) \cdot v = \nabla f(p) \cdot v = 2(p \cdot v) - 2(p \cdot u)(u \cdot v)$$

or, rather,

$$f'(p) = 2p - 2(p \cdot u)u$$

□