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Lemma 1. *The Riemann-Stieltjes integral is a linear operator on the integrator. More formally (e.g., as in Apostol) let* $f \in R(\alpha)$ *and* $f \in R(\beta)$ *on* $[a,b]$*. Then* $f \in R(c_1\alpha + c_2\beta)$ *for* $c_1, c_2 \in \mathbb{R}$*, and*

$$\int_a^b f\ d(c_1\alpha + c_2\beta) = c_1 \int_a^b f\ d\alpha + c_2 \int_a^b f\ d\beta$$

Proof. Let $\gamma = c_1\alpha + c_2\beta$. Write $S(p, f, \gamma) = \sum_1^n f_k \Delta\gamma_k$ and use the definition of γ_k:

$$\begin{aligned}\sum_1^n f_k \Delta\gamma_k &= \sum_1^n f_k \Delta(c_1\alpha + c_2\beta)_k \\ &= c_1 \sum_1^n f_k \Delta\alpha_k + c_2 \sum_1^n f_k \Delta\beta_k \\ &= c_1 S(p, f, \alpha) + c_2 S(p, f, \beta)\end{aligned}$$

Now we must construct “fine enough” partitions p such that these sums approach their respective Riemann-Stieltjes integrals. Given $\varepsilon > 0$, let $p'_\varepsilon \subset p$ such that

$$\left| S(p, f, \alpha) - \int_a^b f\ d\alpha \right| < \varepsilon$$

for all p finer than p'_ε. Similarly, let $p''_\varepsilon \subset p$ such that

$$\left| S(p, f, \beta) - \int_a^b f\ d\beta \right| < \varepsilon$$

for all p finer than p''_ε. To construct a partition finer than both p'_ε and p''_ε, let $p_\varepsilon = p'_\varepsilon \cup p''_\varepsilon$ and then take $p \supset p_\varepsilon$, so that combining the above inequalities gives

$$\left| S(p, f, \gamma) - c_1 \int_a^b f\ d\alpha - c_2 \int_a^b f\ d\beta \right| \leq (|c_1| + |c_2|)\varepsilon$$

□

This property is used extensively in probability theory (integrating with respect to piecewise CDFs) and physics (combinations of discrete and continuous mass density functions).